(Cont'd)

Since $f_n' \rightrightarrows g$, Cauchy criterion for uniform convergence (Thm 8.1.10)

implies $\forall \varepsilon > 0$, $\exists$ $H_1 = H(\frac{\varepsilon}{2(b-a)}) \in \mathbb{N}$ such that

$$\| f_m' - f_n' \|_I < \frac{\varepsilon}{2(b-a)}, \quad \forall m, n \geq H_1$$

Since $(f_n(x_0))$ converges, Cauchy criterion for convergence of sequence (Thm 3.5.5) implies

$\forall \varepsilon > 0$, $\exists$ $H_2 = H(\frac{\varepsilon}{2}) \in \mathbb{N}$ such that

$$| f_m(x_0) - f_n(x_0) | < \frac{\varepsilon}{2}, \quad \forall m, n \geq H_2$$

Hence using $(*)_1$,

$\forall \varepsilon > 0$, $\exists$ $H = \max\{H_1, H_2\} \in \mathbb{N}$ such that if $m, n \geq H$,

$$\| f_m - f_n \|_I < \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon$$

Then Cauchy Criterion for uniform convergence again implies

$f_m \rightrightarrows f$ for some function $f : I \to \mathbb{R}$

(converges uniformly to some f)

Next, we need to show that f is differentiable and

$f' = g$.

Let $c \in I$, then mean value thm $\Rightarrow$ for $x \in I$ & $x \neq c$

$$(f_m - f_n)(x) - (f_m - f_n)(c) = (f_m' - f_n')(z)(x-c)$$ for some z between x & c.

$$\therefore \left| \frac{f_m(x) - f_m(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| = |f_m'(z) - f_n'(z)|$$

$$\leq \|f_m' - f_n'\|_I$$

Hence $\forall \varepsilon > 0$,

$$\left| \frac{f_m(x) - f_m(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| < \frac{\varepsilon}{2(b-a)} \quad \text{for } m, n \geq H_1$$

letting $m \to \infty$ and using $f_m \rightrightarrows f$, we have for $x \neq c$

$$\left| \frac{f(x) - f(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| \leq \frac{\varepsilon}{2(b-a)} \quad \text{for } n \geq H_1$$

Now using $f_n' \rightrightarrows g$ again

for the same $\varepsilon > 0$, $\exists N = N(\varepsilon) \in \mathbb{N}$ s.t.

$$|f_n'(c) - g(c)| < \varepsilon \quad \text{for } n \geq N$$

Then let $K = \max\{H_1, N\} \in \mathbb{N}$

$$\left| \frac{f(x) - f(c)}{x-c} - g(c) \right| \leq \left| \frac{f(x) - f(c)}{x-c} - \frac{f_K(x) - f_K(c)}{x-c} \right|$$

$$+ \left| \frac{f_K(x) - f_K(c)}{x-c} - f_K'(c) \right| + |f_K'(c) - g(c)|$$

$$< \left(1+\frac{1}{2(b-a)}\right)\varepsilon + \left|\frac{f_K(x)-f_K(c)}{x-c} - f_K'(c)\right|$$

Note that for the same $\varepsilon>0$, $\exists\, \delta_{\varepsilon,c}>0$ such that

$$\left|\frac{f_K(x)-f_K(c)}{x-c} - f_K'(c)\right| < \varepsilon, \quad \text{if } |x-c|<\delta_{\varepsilon,c}\ (x\neq c).$$

Therefore, we have proved that $\forall \varepsilon>0$, $\exists\, \delta_{\varepsilon,c}>0$ s.t.

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| < \left(2+\frac{1}{2(b-a)}\right)\varepsilon \text{ provided } |x-c|<\delta_{\varepsilon,c}.$$

Since $\varepsilon>0$ is arbitrary,

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \text{ exists \& equals } g(c).$$

As $c\in I$ is arbitrary, f is differentiable on I and

$$f' = g. \quad ※$$

<u>Interchange of Limit and Integral</u>

<u>Thm 8.2.4</u> Let
- $f_n \in R[a,b]$ for $n=1,2,3,\ldots$ (Riemann integrable)
- $f_n \rightrightarrows f$ on $[a,b]$ (converges uniformly on $[a,b]$ to f)

Then $f\in R[a,b]$ and $\lim_{n\to\infty}\int_a^b f_n = \int_a^b f$

(i.e. f_n converges <u>uniformly</u> $\Rightarrow$ $\lim_{n\to\infty}\int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n$)

Pf: By Cauchy Criterion for Uniform Convergence (Thm 8.1.10),

$\forall \varepsilon > 0$, $\exists H(\varepsilon) > 0$ s.t.

if $m > n \geq H(\varepsilon)$, then $\|f_m - f_n\|_{[a,b]} < \varepsilon$

i.e. $-\varepsilon < f_m(x) - f_n(x) < \varepsilon \quad \forall x \in [a,b]$ —— $(*)_1$

Hence $-\varepsilon(b-a) \leq \int_a^b f_m - \int_a^b f_n \leq \varepsilon(b-a)$

i.e. $\left|\int_a^b f_m - \int_a^b f_n\right| \leq \varepsilon(b-a)$

Since $\varepsilon > 0$ is arbitrary, this implies the seq. of numbers $\left(\int_a^b f_n\right)$ is a Cauchy sequence.

$\therefore \lim_{n\to\infty} \int_a^b f_n = A$ exists, (denoted by A).

$\Rightarrow \forall \varepsilon > 0$, $\exists K(\varepsilon) > 0$ s.t.

$\left|\int_a^b f_n - A\right| < \varepsilon$, for $n \geq K(\varepsilon)$. —— $(*)_2$

And letting $m \to \infty$ in $(*)_1$, we have

$\forall \varepsilon > 0$, $\exists H(\varepsilon) > 0$ s.t. if $n \geq H(\varepsilon)$, then

$-\varepsilon \leq f(x) - f_n(x) \leq \varepsilon$

i.e. $|f_n(x) - f(x)| \leq \varepsilon \quad \forall x \in [a,b]$ —— $(*)_3$

Now, let $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{l}$ be a tagged partition of $[a,b]$.

If $n \geq \max\{H(\varepsilon), K(\varepsilon)\}$, we have

$$\left|S(f_n;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})\right| = \left|\sum_{i=1}^{\ell} f_n(t_i)(x_i - x_{i-1}) - \sum_{i=1}^{\ell} f(t_i)(x_i - x_{i-1})\right|$$

$$= \left|\sum_{i=1}^{\ell}(f_n(t_i) - f(t_i))(x_i - x_{i-1})\right|$$

$$\leq \sum_{i=1}^{\ell}|f_n(t_i) - f(t_i)|(x_i - x_{i-1})$$

$$\leq \varepsilon \sum_{i=1}^{\ell}(x_i - x_{i-1}) \qquad (\text{by } (*)_3)$$

$$= \varepsilon(b-a)$$

Then

$$|S(f;\dot{\mathcal{P}}) - A| \leq |S(f;\dot{\mathcal{P}}) - S(f_n;\dot{\mathcal{P}})| + |S(f_n,\dot{\mathcal{P}}) - A|$$

$$\leq \varepsilon(b-a) + \left|S(f_n;\dot{\mathcal{P}}) - \int_a^b f_n\right| + \left|\int_a^b f_n - A\right|.$$

$$< \varepsilon(b-a+1) + \left|S(f_n;\dot{\mathcal{P}}) - \int_a^b f_n\right| \qquad (\text{by } (*)_2)$$

Finally, fix an $n_0 \geq \max\{H(\varepsilon), K(\varepsilon)\}$ and using $f_{n_0} \in \mathcal{R}[a,b]$, $\exists\ \delta_{\varepsilon,n_0} > 0$ (depends on n_0 too) s.t.

if $\|\dot{\mathcal{P}}\| < \delta_{\varepsilon,n_0}$, then $\left|S(f_{n_0};\dot{\mathcal{P}}) - \int_a^b f_{n_0}\right| < \varepsilon$.

Hence $\forall \varepsilon > 0$, if $\|\dot{\mathcal{P}}\| < \delta_{\varepsilon,n_0}$, we have

$$\left|S(f;\dot{\mathcal{P}}) - A\right| < \varepsilon(b-a+1) + \varepsilon = \varepsilon(b-a+2).$$

Since $\varepsilon > 0$ is arbitrary, we have proved that

$$f \in R[a,b] \text{ and } \int_a^b f = A = \lim_{n\to\infty} \int_a^b f_n . \quad ⌘$$

<u>Thm 8.2.5</u> (<u>(Uniform) Bounded Convergence Theorem</u>)

Let
- $f_n \in R[a,b]$ $\forall n = 1, 2, 3, \cdots$ (Riemann integrable)
- $f_n \to f$ on $[a,b]$ (pointwise convergence)
- $f \in R[a,b]$
- $\exists B > 0$ such that $\|f_n\|_{[a,b]} \le B$, $\forall n = 1, 2, 3 \cdots$
  (i.e. $|f_n(x)| \le B$, $\forall x \in [a,b]$ & $\forall n = 1, 2, 3, \cdots$)

Then $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n$

<u>Pf</u>: Omitted

Remark: The condition in Bounded Convergence Thm is weaker than Thm 8.2.4

<u>Thm 8.2.6</u> (<u>Dini's Theorem</u>)

Let
- $f_n : [a,b] \to \mathbb{R}$ be a <u>monotone</u> seq. of <u>continuous</u> functions
- $f_n \to f$ on $[a,b]$ (pointwise convergence)
- f is <u>continuous</u>

Then $f_n \rightrightarrows f$ on $[a,b]$ (uniform convergence)

Remark:

monotone $\begin{cases} \text{increasing seq.: } n \le m \Rightarrow f_n(x) \le f_m(x), \forall x \in [a,b] \\ \text{decreasing seq.: } n \le m \Rightarrow f_n(x) \ge f_m(x), \forall x \in [a,b] \end{cases}$

Pf We assume f_n is a decreasing seq. The proof is similar for increasing sequence.

let $g_n = f_n - f$.

Then $g_n \ge 0$ decreasing, continuous, and

$g_n \to 0$ (pointwise)

(Different proof from the Textbook)

Assume on the contrary that $g_n \not\rightrightarrows 0$ (not uniform).

Then by lemma 8.1.5,

$\exists \varepsilon_0 > 0$, a subseq g_{n_k} of g_n, and a seq $x_k \in [a,b]$

s.t. $|g_{n_k}(x_k) - 0| \ge \varepsilon_0$

$\Rightarrow \quad g_{n_k}(x_k) \ge \varepsilon_0$

Since $x_k \in [a,b]$, (x_k) is a bounded seq.

Then Bolzano-Weierstrass Thm (Thm 3.4.8) implies that

x_k has a convergence subseq $(x_{k_\ell})_{\ell=1}^{\infty}$

Let $\lim_{\ell \to \infty} x_{k_\ell} = z$.

Since $[a,b]$ is a closed interval, $z \in [a,b]$.

By assumption $g_n(z) \to 0$ as $n \to \infty$.

$\therefore$ $g_{n_{k_\ell}}(z) \to 0$ as $\ell \to \infty$.

$\Rightarrow$ $\exists\, L > 0$ s.t.

if $\ell \geq L$, then $0 \leq g_{n_{k_\ell}}(z) < \frac{\varepsilon_0}{2}$

In particular $0 \leq g_{n_{k_L}}(z) < \frac{\varepsilon_0}{2}$

For clarity of presentation, denote n_{k_L} by N.

Then $0 \leq g_N(z) < \frac{\varepsilon_0}{2}$.

Now using continuity of g_N $(= g_{n_{k_L}})$

$$\lim_{\ell\to\infty} g_N(x_{k_\ell}) = g_N(z) \qquad \left(\text{since } \lim_{\ell\to\infty} x_{k_\ell} = z\right)$$

$\Rightarrow$ $\exists\, L_1 > 0$ s.t. if $\ell \geq L_1$, then

$$g_N(x_{k_\ell}) < \frac{\varepsilon_0}{2}$$

Using the assumption that g_n is decreasing, we have

$$g_n(x_{k_\ell}) \leq g_N(x_{k_\ell}) < \frac{\varepsilon_0}{2}, \quad \forall\, n \geq N = n_{k_L}$$

In particular, for $n = n_{k_\ell}$ with $\ell \geq \max\{L, L_1\}$, we have

$$\varepsilon_0 \leq g_{n_{k_\ell}}(x_{k_\ell}) \leq \frac{\varepsilon_0}{2}$$

which is a contradiction.

Therefore $g_n \rightrightarrows 0$ (uniform convergence) $\#$

<u>Remark</u>: The approach in Textbook requires the fact that for any given function $t \mapsto \delta(t) > 0$ on $[a,b]$ $\exists$ finitely many $t_i \in [a,b]$, $i=1,\cdots,l$ such that

$$[a,b] \subset \bigcup_{i=1}^{l} (t_i - \delta(t_i),\ t_i + \delta(t_i)).$$

This needs Thm 5.5.5 which is not covered in MATH2050.

These two proofs use different versions of the fact that $[a,b]$, a closed & bounded interval, is <u>compact</u>:

(i) Any sequence in $[a,b]$ has a subsequence converges to some point in $[a,b]$.

(ii) For any open cover of $[a,b]$, $[a,b] \subset \bigcup_{\lambda} (\alpha_\lambda, \beta_\lambda)$, (where $(\alpha_\lambda, \beta_\lambda)$ are open intervals, could be infinitely many) has finite subcover, i.e.

$\exists$ finitely many λ_i, $i=1,\cdots,l$ such that

$$[a,b] \subset \bigcup_{i=1}^{l} (\alpha_{\lambda_i}, \beta_{\lambda_i})$$

Detail discussion and proof are skipped.

§8.3 The Exponential and Logarithmic Functions

The Exponential Function

Thm 8.3.1 $\exists$ a function $E: \mathbb{R} \to \mathbb{R}$ s.t.

(i) $E'(x) = E(x)$, $\forall x \in \mathbb{R}$

(ii) $E(0) = 1$

Pf: Let $E_1(x) = 1 + x$

$$E_2(x) = 1 + \int_0^x E_1 = 1 + \int_0^x (1+t)\,dt = 1 + x + \frac{x^2}{2}$$

$$\vdots$$

$$E_{n+1}(x) = 1 + \int_0^x E_n, \quad \forall n = 1, 2, 3 \cdots$$

Then "Induction" implies for all $n = 1, 2, 3, \cdots$

$$E_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \qquad (\text{Ex!})$$

Consider a closed interval $[-A, A]$. $(A > 0)$

Then for $x \in [-A, A]$ and $m > n > 2A$, we have

$$|E_m(x) - E_n(x)| = \left| \frac{x^{n+1}}{(n+1)!} + \cdots + \frac{x^m}{m!} \right|$$

$$\leq \frac{A^{n+1}}{(n+1)!} + \cdots + \frac{A^m}{m!} \qquad (\text{since } |x| \leq A)$$

$$\leq \frac{A^{n+1}}{(n+1)!} \left(1 + \frac{A}{n+2} + \cdots + \frac{A^{m-n-1}}{m(m-1)\cdots(n+2)} \right)$$

$$\leq \frac{A^{n+1}}{(n+1)!}\left(1+\frac{A}{n}+\cdots+\frac{A^{m-n-1}}{n^{m-n-1}}\right)$$

$$\leq \frac{A^{n+1}}{(n+1)!}\left[1+\frac{1}{2}+\cdots+\left(\frac{1}{2}\right)^{m-n-1}\right] \quad (\text{since } n>2A)$$

$$< \frac{2A^{n+1}}{(n+1)!}$$

Taking sup. over $[-A,A]$, we have $\forall\, m>n>2A$

$$\|E_m - E_n\|_{[-A,A]} \leq \frac{2A^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as } n\to\infty$$

Cauchy Criterion for Uniform Convergence (Thm 8.1.10) implies

$E_n(x)$ converges uniformly to some function on $[-A,A]$

Since $A>0$ is arbitrary, we conclude that

$E_n(x)$ converges for all $x\in\mathbb{R}$ (not necessary uniform on $\mathbb{R}$)

It is because, $\forall\, x\in\mathbb{R}$, we can find an $A>0$ s.t.

$x\in[-A,A]$. Then the uniform convergence on $[-A,A]$

implies $E_n(x)$ converges.

Denote the (pointwise) limit by

$$E(x) \stackrel{\text{denote}}{=\!=} \lim_{n\to\infty} E_n(x), \quad \forall\, x\in\mathbb{R}.$$

Note that $\quad E_n(x) = 1+\int_0^x E_{n-1}$

$$\Rightarrow \quad E_n(0)=1, \quad \forall\, n=2,3,\cdots \quad (E_1(0)=1 \text{ is clear})$$

Hence $\quad E(0) = \lim_{n\to\infty} E_n(0) = 1.$

Also by Fundamental Thm of Calculus (2nd Form) Thm 7.3.5

and $$E_n(x) = 1 + \int_0^x E_{n-1},$$

we have $$E_n'(x) = E_{n-1}(x)$$

$\therefore\ \forall A>0,$

$$\left(E_n\big|_{[-A,A]}\right)' = E_{n-1}\big|_{[-A,A]} \rightrightarrows E\big|_{[-A,A]} \text{ (uniform)}$$

Then by Thm 8.2.3, together with $E_{n+1}\big|_{[-A,A]}(0) \to E(0)$

we have $E\big|_{[-A,A]}$ is differentiable and

$$\left(E\big|_{[-A,A]}\right)' = E\big|_{[-A,A]}$$

Since $A>0$ is arbitrary, this implies $E'(x)$ exists $\forall x\in\mathbb{R}$ and

$$E'(x) = E(x)$$

Cor 8.3.2 The function E has derivative of <u>every order</u> and
$$E^{(n)}(x) = E(x), \quad \forall x \in \mathbb{R}.$$

Pf: Easy, by induction.

$\boxed{\text{Cor 8.3.3} \quad \text{If } x>0, \text{ then } E(x) > 1+x}$

Pf: From $E_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$, we have

$$m > n \Rightarrow E_m(x) > E_n(x), \quad \forall x > 0$$

Letting $m \to \infty$, and take $n > 1$, we have

$$E(x) \geq E_n(x) > E_1(x) = 1 + x, \quad \forall x > 0 \quad \#$$

Thm 8.3.4: $E: \mathbb{R} \to \mathbb{R}$ is the <u>unique</u> function satisfying

$$(*) \begin{cases} E'(x) = E(x), & \forall x \in \mathbb{R} \\ E(0) = 1 \end{cases}$$

Pf: Suppose that E_1 & E_2 satisfy $(*)$.

Let $F = E_1 - E_2$.

Then F is differentiable and

$$\begin{cases} F' = E_1' - E_2' = E_1 - E_2 = F \\ F(0) = E_1(0) - E_2(0) = 0 \end{cases}$$

Moreover, induction $\Rightarrow$ F has derivatives of every order

and $\quad F^{(n)} = F, \quad \forall n = 1, 2, 3, \cdots$

Hence $\quad F^{(n)}(0) = F(0) = 0, \quad \forall n = 1, 2, 3, \cdots$

Applying Taylor's Thm 6.4.1 to $F|_{[0,x]}$ for $x>0$

or $F|_{[x,0]}$ for $x<0$,

we have for $x>0$

$$F(x) = F(0) + F'(0)x + \cdots + \frac{F^{(n-1)}(0)}{(n-1)!}x + \frac{F^{(n)}(c_n)}{n!}x^n$$

$$= \frac{F(c_n)}{n!}x^n \quad \text{for some } c_n \in [0,x].$$

Since F is cts on $[0,x]$, F is bdd on $[0,x]$.

$\therefore$ $\exists K>0$ (depends on x) such that

$$|F(c_n)| \le K \quad (\forall n=1,2,\cdots)$$

$$\Rightarrow \quad |F(x)| \le K\frac{x^n}{n!}$$

Since $\lim_{n\to\infty}\frac{x^n}{n!}=0$, letting $n\to\infty$, we have $|F(x)|=0$.

$\therefore$ $F(x)\equiv 0$, $\forall x>0$

Similarly for $x<0$, we also have $F(x)\equiv 0$, $\forall x<0$.

All together $F(x)\equiv 0$.

i.e. $E_1(x)\equiv E_2(x)$

$\therefore$ The function E is unique.

$\#$

Def 8.3.5 The Unique function $E: \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} E'(x) = E(x), \ \forall x \in \mathbb{R} & \text{——— (i)} \\ E(0) = 1 & \text{——— (ii)} \end{cases}$$

is called the <u>exponential function</u> and is denoted by

$$e^x \quad \text{or} \quad \exp(x)$$

The number $e = E(1)$ is called the <u>Euler's number</u>.

Thm 8.3.6 Exponential function E satisfies

- $E(x) \neq 0, \ \forall x \in \mathbb{R}$ ——— (iii)
- $E(x+y) = E(x)E(y) \ \forall x, y \in \mathbb{R}$ ——— (iv)
- $E(r) = e^r, \ \forall r \in \mathbb{Q}$ ——— (v)

<u>Remarks</u>:
- (iv) justifies the use of notation $e^x = E(x)$:

$$e^{x+y} = e^x e^y, \quad \forall x, y \in \mathbb{R}$$

- In (v), "RHS" means the <u>rational power</u> of the number e

<u>Pf</u>: (iii) Suppose on the contrary that $E(\alpha) = 0$ for some $\alpha \in \mathbb{R}$.

Since $E(0) = 1$, $\alpha \neq 0$.

Let J_α = closed interval $[0, \alpha]$ or $[\alpha, 0]$ depends on the sign of α.

and $K > 0$ such that $|E(x)| \leq K, \ \forall x \in J_\alpha$.

As E has derivative of all order, Taylor's Thm 6.4.1

(base at $x_0 = \alpha$) implies $\forall n = 1, 2, 3, \cdots$

$$E(0) = E(\alpha) + \frac{E'(\alpha)}{1!}(0-\alpha) + \cdots + \frac{E^{(n-1)}(\alpha)}{(n-1)!}(0-\alpha)^{n-1}$$

$$+ \frac{E^{(n)}(c_n)}{n!}(0-\alpha)^n \qquad \text{for some } c_n \in J_\alpha .$$

$$\Rightarrow \quad 1 = E(\alpha) + E(\alpha)(-\alpha) + \frac{E(\alpha)}{2!}(-\alpha)^2 + \cdots + \frac{E(\alpha)}{(n-1)!}(-\alpha)^{n-1}$$

$$+ \frac{E(c_n)}{n!}(-\alpha)^n$$

since $E(0) = 1$, and $E^{(k)} = E \quad \forall k = 1, 2, \cdots$

By $E(\alpha) = 0$, $\qquad 1 = \frac{E(c_n)}{n!}(-\alpha)^n$

$$\Rightarrow \qquad 1 \le \frac{K|\alpha|^n}{n!}, \quad \forall n = 1, 2, \ldots$$

$$(\longrightarrow 0 \quad \text{as } n \to \infty)$$

which is impossible. $\therefore E(\alpha) \neq 0, \ \forall \alpha \in \mathbb{R}$.

Pf: (iv) Fix y and consider the ratio

$$G(x) = \frac{E(x+y)}{E(y)} \quad \text{as a function of } x .$$

- $G(x)$ is well-defined since $E(y) \neq 0$ by (iii).

- $G(0) = \frac{E(y)}{E(y)} = 1$.

- E differentiable $\Rightarrow$ G differentiable and

$$G'(x) = \frac{E'(x+y)}{E(y)} \quad \text{(by Chain rule)}$$

$$= \frac{E(x+y)}{E(y)} = G(x) \qquad \text{(by (i))}$$

By Thm 8.3.4, $G(x) = E(x)$, $\forall x \in \mathbb{R}$

$\therefore \quad E(x+y) = E(x)E(y) \quad \forall x, y \in \mathbb{R}.$

Pf: (v) (Ex! MATH1010 level proof) ※

Thm 8.3.7:
- Exponential function E is strictly increasing on $\mathbb{R}$ and
- $E(\mathbb{R}) = \{y \in \mathbb{R} : y > 0\}$.

Further,
$$\left.\begin{array}{l} \bullet \ \lim_{x \to -\infty} E(x) = 0 \\ \bullet \ \lim_{x \to +\infty} E(x) = +\infty \end{array}\right\} \text{—— (vi)}$$

Pf: E differentiable on $\mathbb{R}$ $\Rightarrow$ E continuous on $\mathbb{R}$.

It's proved in (iii) in Thm 8.3.6 that $E(x) \neq 0$, $\forall x \in \mathbb{R}$.

$\therefore \quad E(0) = 1 \Rightarrow E(x) > 0, \quad \forall x \in \mathbb{R}$

Otherwise, intermediate value thm $\Rightarrow$ $E(x_0) = 0$ for some x_0

which is a contradiction.

Hence $E'(x) = E(x) > 0 \quad \forall x \in \mathbb{R}$

which implies E is strictly increasing.

By Cor 8.3.3, $E(x) > 1+x \quad \forall x > 0$

$$\Rightarrow \lim_{x \to +\infty} E(x) = +\infty .$$

Using (iv), if $x<0$, then $E(x) = \dfrac{1}{E(|x|)}$

$$\therefore \lim_{x \to -\infty} E(x) = \lim_{|x| \to +\infty} \frac{1}{E(|x|)} = 0 .$$

Finally, with continuity of E and the values of the limits, intermediate value thm implies

$$\forall y > 0, \ \exists x \in \mathbb{R} \text{ s.t. } y = E(x).$$

Therefore $E(\mathbb{R}) = \{ y \in \mathbb{R} : y > 0 \}$.